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The Impact of The Multiplicative Random Effects on The Error Terms

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Abstract

In this research, we investigate the effect of multiplicative random errors. We build a model which is a physical representation of the multiplicative central limit theorem in mathematical statistics. This theorem demonstrates how the log-normal distribution arises from many minor and multiplicative random effects. A log-normal process is the statistical realization of the multiplicative product of many independent positive random variables. It is always said that the errors in statistical models have a normal distribution, and the reason is that each error component is an effect of an unknown factor. When these components are added together, their sum will have a normal distribution according to the central limit theorem. The total effect of unknown factors is often not additive, and their final effect will be equal to the product of their individual effects. We provide a numerical example and discuss the multiplicative impact on social science.

Keywords: Chaos theory, Error terms, Log-normal distribution, Multiplicative effect, Social Science.

1 | Introduction

Statistical methods generally consider assumptions (e.g., normality in linear regression models). The violations of the assumptions can cause various issues, such as statistical errors and biased estimates. The impact of such issues can also change from inconsequential to critical. Accordingly, it is of great importance to take proper assumptions in to account when employing statistical methods. In the prior literature, however, assumptions have not considered appropriately. Shatz (2023) presented a prevalent but problematic approach to diagnostics-testing assumptions using null hypothesis significance tests. He also consolidated and illustrated some issues faced by this approach using simulations. Nimon (2012) stated that the validity of inferences drawn from statistical test results depend on how data meet associated assumptions.

This study considers a model that is a physical representation of the multiplicative central limit theorem in mathematical statistics. The central limit theorem shows minor and multiplicative random effects on the log-normal distribution (Choi, 2016). The errors in statistical models have a normal distribution, since each error component is an effect of an unknown factor. The sum of error elements has a normal distribution based on the central limit theorem. The final effect of unknown factors is not additive, and the final effect will be equal to the product of their individual effects. For example,

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if a_i is the effect of factor i, then total effects is equal to $|a| = \prod |a_i|$. Thus, $log(|a|) = \sum \log(|a_i|)$ in which $log(|a|)$ follows a normal distribution according to the central limit theorem, and $|a|$ follows a log-normal distribution. In the other words, the distribution of error elements should be a log-normal distribution. Since the logarithm of each error element is summed, and according to the central limit theorem, the logarithm of the error elements' sum will have a normal distribution. Hence, the error element will have a log-normal distribution.

In the probability theory, a log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed. Thus, if the random variable X is log-normally distributed, then $Y = \ln(X)$ has a normal distribution. Equivalently, if Y has a normal distribution, then the exponential function of Y, $X = \exp(Y)$, has a log-normal distribution. A random variable, which is log-normally distributed, takes only positive real values. It is a useful model for measurements in exact and engineering sciences, medicine, economics, and other topics (Johnson et al., 1994).

According to the central limit theorem, a log-normal process is the statistical realization of the multiplicative product of many independent positive random variables. The log-normal distribution is important in describing natural phenomena. Many natural growth processes are driven by the accumulation of many small percentage changes which become additive on a log scale. Therefore, the distribution of the accumulated changes will be well approximated by a log-normal based on the "Multiplicative Central Limit Theorem". In addition to the central limit theorem, the observations about the fundamental natural laws imply the multiplications and divisions of positive variables. For example, the simple gravitation law connects masses and distance with the resultant force. Hence, considering the log-normal distributions of variables leads to consistent models (Sutton, 1997).

2. Problem definition

The normality assumption in a regression model is often reasonable based on the central limit theorem. The "Central Limit Theorem for Sums" states that the sum of many independent variables approximately has a normal distribution, even if each independent variable follows different distributions. This theorem can be applied to a linear regression model. We know that if we iteratively obtain different samples with size n, then rerun the linear regression model, and compute the error value for the same X, we likely have different error values. Thus, conditional on $X = X_i$, we can observe $\mathcal{E}_i$ (error term) as the sum of many other independent errors from omitting the important variable. Each of these other independent errors follows an unknown distribution. Therefore, based on the central limit theorem for Sums, the sampling distribution of the individual error term ($\mathcal{E}_i$) is a normal distribution. This is due to the sources of many other independent errors influencing a single observation (Zhu, 2022). However, when the final effect of other independent errors is not additive and it has a multiplicative effect, then the central limit theorem for Sums cannot be applied and it is proven that the error terms in this case follow a log-normal distribution. In this research, we aim to analyze the effect of the normality and log-normality assumption on the results of statistical models like regression.

Although practically it is considered that error terms have the non-normal distribution (Zeckhauser and Thompson, 1970), some essential issues are not explored in the prior literature. Thus, this study contributes to the literature by investigating not additive error terms with log-normal distribution. Moreover, we consider that the final effects of the error terms are multiplicative. Considering such assumptions, we should use statistical models (e.g., maximum likelihood estimation method) should be adapted and resolved them.

Overall, the main contribution of this study is to examine the effects of different distribution on error terms using a simple regression model. We indicate that the results do not coincide with each other and all statistical models should be reformulated based on log-normal error terms with the not-additive effect of random error terms.

The main benefit of this research is to indicate that the results of statistical methods cannot be applied in real situations, as they are not optimal solutions. Because always assuming that error terms have a normal distribution is not correct. This study can help practitioners identify why classical statistical models have poor performance.

As the theoretical point of view, if assuming additive error terms is not verified, then it is concluded that the results of all classical methods applied in a well-known software like Minitab cannot be applicable. Hence, other related software should consider the additive error terms for specific problems rather than considering the classical assumption (i.e., normal error terms).

3. The regression formula under the normal and log-normal errors

Example. Suppose there are two points: (1) $x_1 = 2, y_1 = 5$ and (2) $x_2 = 3, y_2 = 7$. We aim to fit a line with the formula $y = ax$ and these two points. If we assume that the errors have a standard normal distribution, then the following is obtained.

$$y = ax + \mathcal{E} \quad \Rightarrow L = Likelihood = f(\mathcal{E}_1)\, f(\mathcal{E}_1) = \frac{1}{\sqrt{2\,\pi}} e^{\frac{-(y_1 - ax_1)^2}{2}} \frac{1}{\sqrt{2\,\pi}} e^{\frac{-(y_2 - ax_2)^2}{2}}$$

Maximizing the likelihood function leads to the following result.

$$\frac{\partial L}{\partial a} = 0 \quad \Rightarrow \quad \frac{\partial Ln\, L}{\partial a} = 0 \quad \Rightarrow \quad x_1(y_1 - ax_1) + x_2(y_2 - ax_2) = 0$$
$$\Rightarrow \quad a = \frac{x_2 y_2 + x_1 y_1}{{x_2}^2 + {x_1}^2} = \frac{21 + 10}{9 + 4} = \frac{31}{13}$$

If we assume that the errors have a standard log-normal distribution, the following result is obtained.

$$y = ax + \mathcal{E} \;\Rightarrow\; Ln\,\mathcal{E} = N(0\,,1) \;\Rightarrow\; y - ax = \mathcal{E} \;\Rightarrow\; Ln(\,y - ax) = N(0\,,1)$$
$$\Rightarrow L = Likelihood = f(Ln\,\mathcal{E}_1) f(Ln\,\mathcal{E}_2)$$
$$= \frac{1}{\sqrt{2\,\pi}} e^{\frac{-(Ln(y_2 - ax_2))^2}{2}} \frac{1}{\sqrt{2\,\pi}} e^{\frac{-(Ln(y_1 - ax_1))^2}{2}}$$

Thus, according to the maximum likelihood estimation method, we have:

$$\frac{\partial L}{\partial a} = 0 \;\Rightarrow\; \frac{\partial Ln\, L}{\partial a} = 0 \;\Rightarrow\; \frac{\partial([Ln(y_2 - ax_2)]^2 + [Ln(y_1 - ax_1)]^2)}{\partial a} = 0$$

which leads to the following formula.

$$[\frac{x_2}{y_2 - ax_2} Ln\,(y_2 - ax_2) + \frac{x_1}{y_1 - ax_1} Ln\,(y_1 - ax_1)] = 0$$
$$\Rightarrow [(y_2 - ax_2)]^{\frac{x_2}{y_2 - ax_2}} [(y_1 - ax_1)]^{\frac{x_1}{y_1 - ax_1}} = 1 \quad \Rightarrow \quad a = 2$$

The error terms can be negative. If we assume $\mathcal{E}_2$ is negative, then the following result is obtained.

$$Likelihood = f\big(Ln(\,\mathcal{E}_1)\big) f(Ln\,(-\mathcal{E}_2)) = \frac{1}{\sqrt{2\,\pi}} e^{\frac{-\left(Ln(-(y_2 - ax_2))\right)^2}{2}} \frac{1}{\sqrt{2\,\pi}} e^{\frac{-(Ln(y_1 - ax_1))^2}{2}}$$

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$$\frac{\partial L}{\partial a}=0 \;\Rightarrow\; \frac{\partial Ln\,L}{\partial a}=0 \;\Rightarrow\; \frac{\partial\left([Ln(-(y_2-ax_2))]^2+[Ln(y_1-ax_1)]^2\right)}{\partial a}=0$$

which simplifies to the following equation.

$$[\frac{x_2}{y_2-ax_2}Ln\left(-(y_2-ax_2)\right)+\frac{-x_1}{y_1-ax_1}Ln\,(y_1-ax_1)]\;=0$$

$$\Rightarrow [(-y_2+ax_2)]^{\frac{x_2}{y_2-ax_2}}[(y_1-ax_1)]^{\frac{-x_1}{y_1-ax_1}}=1$$

If we assume $\mathcal{E}_1$ is negative, then the following equation is obtained.

$$Likelihood=f\left(Ln(-\mathcal{E}_1)\right)f\left(Ln(\mathcal{E}_2)\right)\rightarrow[(y_2-ax_2)]^{\frac{-x_2}{y_2-ax_2}}[-(y_1-ax_1)]^{\frac{x_1}{y_1-ax_1}}=1$$

The results obtained above indicate different outcomes under various distributions of errors. To be specific, the log-normal error terms achieve $a=2$ and the normal error terms obtain $a=\frac{31}{13}$. This result reveals the importance of the error elements. Specifically, one factor can highly decrease or increase the total amount of errors or even reverse the effect of each error element, since the effect of the error element is multiplicative. As a result, the multiplicative effect of error elements can be considered the same as the concept of the butterfly effect. Because under the butterfly effect, a small event can lead to huge events. To be specific, the effect of a small event is multiplied by other events which can consequently change the final result. Hence, the solution of the problem will completely change with a small event.

4. The effects of multiplicative events on social issues

It is usually said that being good and being bad will have a cost. Hence, people should choose the appropriate morals and behavior in different situations considering this cost. However, based on human and religious values, it is said that being good is the best strategy for life, as this strategy maximizes the people's findings during life and people achieve the optimal point of their life. Conversely, according to the concept of multiplicative effects, a negative effect can change the optimal point of life to the worst point. Thus, based on this concept, a completely immoral life can be optimal in some cases with low probability. As a result, being moral will not always be the optimal policy for people. Therefore, contrary to the popular belief, people should choose a point between the best and the worst option to successfully manage unforeseen events.

The neutral point in mathematics and game theory may be considered as a balance point. As a remark, the balance point is highly important and critical in social issues. Because balance in society is one of the foundations of social power, thus, it is essential to maintain or even improve the balance equilibrium. Based on the multiplicative concept regarding the effects of events, if events reduce the balance power, then the balance power will decrease exponentially. Because the effect of the events is not added with the effect of the balance point, and these two effects are multiplied with each other.

If negative beliefs are common in the society, then such beliefs can lead to occurring events, which reduce the power of balance and accordingly, the society will suffer from disorder and imbalance. The imbalance will lead to occurring more such events and deteriorating the social foundations rapidly. Thus, in such a situation, the best strategy is to quickly discover the source of disorder and adopt effective strategies in accordance with the new facts to deal with false beliefs (Fallahnezhad, 2023).

According to the concept of stability and resilience, dominant social thoughts should play a key role in designing the equilibrium point of the society. Therefore, it is necessary to continuously collect

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information about dominant thoughts in the society, examine the collected information, discover false strategies, and adopt new strategies to improve the balance point and increase social power. For this purpose, we can define societal indicators, which are regularly updated based on new information. We can check the trend of such indicators over time which may result in identifying the out-of-control point quickly. As a result, we can understand which incident or event led to the out-of-control situations of society.

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5. The chaos theory and multiplicative effects

The chaos theory states that the series of events, which we think they are randomly happened, in fact, they are happened according to a certain order, however, the order rules of events may not be known to us. It introduces a new complex concept implying that there is an extremely complex order in the real-world. To be specific, this new concept states that the existing balance is the result of highly complex processes (Fallahnezhad, 2018).

If the concept of multiplicative effect of events is combined with the concept of the chaos theory, a highly complex problem may arise. It means that many events, which we think they are happened by chance, they are occurred based on a specific order and they affect the entire human achievements. Therefore, humans must face such a complex order to design defense systems based on deep intellectual inferences obtained from experience and beliefs. However, every human being behaves inappropriately towards such a complex order, no matter how wise and intelligent he is. Perhaps the reason that man has always sought to rely on a supernatural power throughout history is that this need has been highly necessary and vital for his destiny.

In fact, the reason why people feel they need such concepts even in the modern world is that there are incomplete truths and facts, which are not understood by the science. However, for example, holy books have an eternal impact on human societies and human fate. This concept is against itself in some cases and causes problems, due to the presence of complex and mysterious insights in the holy books, which leads to naive perceptions of the concept.

Another concept that should be mentioned in this context is the law of nature. The law of nature can be regarded the same as the law of conflict for survival that the stronger animal kills and eats the weaker one to survive. This law exists in different forms in human societies, such as theft or murder. As man is an intelligent being, the law of conflict for survival can be interpreted that a man with the knowledge weapon can eliminate an ignorant man and overcome him. If this concept is integrated with the concept of the chaos theory, it means that an ignorant person can easily be deceived when he faced with apparently simple human behaviors, which are as a result of complex actions of a person having science weapon. In such a case, irreparable damages can cause to an ignorant person similar to what happens among animals. The human being emotions can even trap the prey, which happens only among humans. Therefore, if an intelligence person with science weapon behaves according to the law of nature, he can subtly commit various sins and crimes, causing damages. Accordingly, it if great importance to make highly strict laws to control unruly people. As a remark, these laws can be derived from religion or they can be changed according to the society conditions.

The equivalent concept of the multiplicative theory is Bayesian networks where the probabilities of different events are multiplied together and hence each event can be the outcome of different events interacted. Thus, every event we see is formed many factors, which may not be yet identified. According to the concept, it can be concluded that human being must rely on an unlimited power to be able to choose his own path in such infinitely complex society. Because human being ability to understand facts is so limited. In addition, there are many ambiguities even in understanding the simplest concepts. Also, human being cannot bear life crises, and accordingly, the human fate may be tragic and sad.

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6. Conclusion

In this study, the multiplicative effect of events was investigated. In addition, this study showed that the multiplicative effect can completely influence the results of the previous models. We also indicate that ignoring such effect may lead to wrong solutions. This study further discussed that the multiplicative effect of events can result in deep social crises, which is equivalent to the concept of the butterfly effect. This study can be extended by investigating the combination of additive and multiplicative effects.

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